Tick your teacher
o Miss Cheng
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o Ms Rimando
o Miss Sindel

PERTH MODERN SCHOOL

YR11 MATHEMATICS SPECIALIST – 2018

TEST 1 – Reasoning (8%)

PERTH MODERN SCHOOL
Exceptional schooling Exceptional students

NAME: Solutions **DATE:** Monday 26/02/18 7:45am

[To achieve full marks and to allow assessment of outcomes, working and reasoning should be shown.]
[A maximum of 2 marks will be deducted for incorrect rounding, units, etc.]

This is a *Calculator Free Assessment – 45 minutes / 38 marks*

1. [6 marks = 2, 2, 2]
Determine whether each of the following statement is true/false. Prove in general if the statement is true; disprove the false statements using counter-example(s).

(a) The sum of 3 consecutive whole numbers is divisible by 3.

True. Let the three numbers be $x-1$, x, $x+1$

$(x-1)+(x)+(x+1) = 3x$

which is divisible by 3.

(b) For any real number x, if x^2 is an odd number, then x must be an odd number.

False. E.g. $x^2 = 3$

$x = \sqrt{3}$ not an odd number.

(c) If a number is a multiple of m, and it is also a multiple of n, then it is a multiple of mn.

False. 24 is a multiple of 6, also a multiple of 8.

but is not a multiple of 48.

2. [3 marks]
If n is an integer, prove that $n + n^2$ is always even.

$n+n^2 = n(1+n)$

n and $(n+1)$ are two consecutive integers,

so one of them must be even.

therefore the product must be even.

3. Prove the following inequality **[4 marks]**
$\frac{a}{b}+\frac{b}{a}\geq 2$

$\frac{a}{b}+\frac{b}{a}-2$

$= \frac{a^2+b^2-2ab}{ab}$

$= \frac{(a-b)^2}{ab}$

as $(a-b)^2 \geq 0$.

$\therefore \frac{(a-b)^2}{ab} \geq 0$

$\therefore \frac{a}{b}+\frac{b}{a} \geq 2$.

4. [4 marks]
Given that X = 0. 234343434…… Convert X as a fraction.

$10X = 2.343434\ldots$ ✓

$1000X = 234.343434\ldots$ ✓

$990X = 232$ ✓

$X = \frac{232}{990}$ ✓ or $\frac{116}{495}$

5. [6 marks = 3, 3]
Write down the contrapositive of the following. Determine whether each of the contrapositive statements is true. Prove in general if the statement is true; disprove the false statements using counter-example(s).

(a) If a product of two positive real numbers is greater than 100, then at least one of the number is greater than 10.

or equal

If both of two positive real numbers are smaller than 10, then the product is smaller or equal than 100. ✓

True ✓ Let $0 < x \leq 10$, $0 < y \leq 10$ ... (1 mark off for missing "equal")

$0 < xy \leq 100$. ✓

(b) If $a, b \in R$, such that $a > b$, then $a^2 > b^2$.

If $a^2 \leq b^2$ then $a \leq b$ for $a, b \in \mathbb{R}$. ✓

False ✓ E.g. $4 \leq 9$ $(a^2 \leq b^2)$

however, $(-2) > (-3)$. ✓

6. [5 marks]

Use the fact that if n^2 is divisible by 5, then n is divisible by 5, to prove that $\sqrt{5}$ is irrational, using Proof by Contradiction.

Assume $\sqrt{5}$ is rational and therefore can be expressed as $\frac{a}{b}$ where a & b have no common factors. ✓

$$\sqrt{5} = \frac{a}{b}$$

$$5 = \frac{a^2}{b^2}$$

$\therefore a^2 = 5b^2$

i.e a^2 is a multiple of 5 ✓

$\therefore$ a is a multiple of 5

i.e. $a = 5k$

$a^2 = 25k^2$ ✓

$$5 = \frac{25k^2}{b^2}$$

$\therefore b^2 = 5k^2$

$\therefore$ b^2 is a multiple of 5

$\therefore$ b is a multiple of 5 also. ✓

$\therefore$ a and b will have common factor 5.

which contradicts the assumption ✓

$\therefore \sqrt{5}$ is irrational.

7. [5 marks]

Use mathematical induction to prove that $4^{2n} - 1$ is always divisible by 5, for $n \in N$.

$n=1$, $4^2 - 1 = 15$ is divisible by 5.

Assume $n=k$, $4^{2k} - 1 = 5m$. i.e. $4^{2k} = 5m+1$, $m \in \mathbb{Z}$

For $n=k+1$, $4^{2(k+1)} - 1 = 4^{2k+2} - 1$

$= 4^2 \times 4^{2k} - 1$

$= 16 \times (5m+1) - 1$

$= 16 \times 5m + 16 - 1$

$= 16 \times 5m + 15$

$= 5(16m+3)$

which is a multiple of 5.

Hence this is true for $n=k+1$ and is true for $n=1$

therefore is true for $n=k$.

By proof by induction, $4^{2n} - 1$ is always divisible by 5.

8. [5 marks]
The total of adding up numbers that are doubled each time is the next term minus the first term. Verify this rule using proof by induction by proving the following.

For $n \in N, 5 + 10 + 20 + \cdots + 5 \times 2^{n-1} = 5 \times 2^n - 5$

$n=1 \quad 5 = 5 \times 2^1 - 5$ is true. ✓

Assume this is true for $n=k$.

$$5 + 10 + \cdots + 5 \times 2^{k-1} = 5 \times 2^k - 5$$ ✓

For $n=k+1$

$$\left(5 + 10 + \cdots + 5 \times 2^{k-1}\right) + 5 \times 2^k$$

$$= \left(5 \times 2^k - 5\right) + 5 \times 2^k$$ ✓

$$= 5 \times 2^k \times 2 - 5$$

$$= 5 \times 2^{k+1} - 5$$ ✓

Hence, this is true for $n=k+1$. therefore it is true for $n=k$. ✓

By proof by induction, $n \in \mathbb{N}$,

$$5 + 10 + \cdots + 5 \times 2^{n-1} = 5 \times 2^n - 5.$$